

Math 474, Spring 2004

Solutions to Assignment 2

1. For integer a_i let r_i be the remainder when dividing by n , that is $a_i = q_i n + r_i$ for $0 \leq r_i \leq n-1$. Since there are n possible remainders $0 \leq r_i \leq n-1$ (boxes), two of the $n+1$ numbers (objects) must have the same remainder by the PHP. If these numbers are a_i, a_j with $i \neq j$, then $a_i - a_j = (q_i n + r_i) - (q_j n + r_j) = (q_i - q_j)n$ is divisible by n as desired.
2. For person i , let a_i be the number of acquaintances this person has in the group. Since there are n people in the group and nobody is acquainted with themselves we have $0 \leq a_i \leq n-1$. (Can't apply the PHP yet, since n numbers and n boxes!)

Observe that it is impossible to have both $a_i = 0$ and $a_j = n-1$, since either i and j are acquainted (in which case $a_i > 0$) or they are not (in which case $a_j < n-1$). Thus if every number from 1 to $n-2$ gets its own box, but 0 and $n-1$ get the same box, then we have $n-1$ boxes. Since we have n numbers it follows by the PHP that two of them, say a_i, a_j , are in the same box. Thus we must have $a_i = a_j$: This is obvious for the first $n-2$ boxes and for the last box it follows by the observation above.

3. (a) Suppose the square has vertices (x, y) with $x, y \in \{0, 1\}$. By drawing a vertical line $x = 1/2$ and a horizontal line $y = 1/2$ we split the square into four smaller squares. Thus if five points are chosen within the big square of side length 1, then by the PHP two of them must be in the same small square. But the farthest two points can be apart in the small square is if they are diagonal from each other, which means they are at distance $1/\sqrt{2}$ from each other.
- (b) The four vertices of the big square are mutually at distance $1 > 1/\sqrt{2}$ or $\sqrt{2} > 1/\sqrt{2}$.
- (c) Draw 3 lines by pairwise connecting the midpoints of the sides of the big equilateral triangle. This creates 4 small equilateral triangles. Thus if five points are chosen within the big triangle, then by the PHP two of them must be in the same small triangle. But the farthest two points can be apart in the small triangle is if they are vertices, that is at distance $1/2$ from each other.
- (d) Pick the 3 vertices of the big triangle (these are mutually 1 unit apart) and the center point of the triangle. Since in an equilateral triangle the four special lines (altitude, angular bisector, perpendicular bisector, median) are identical, the height of these lines is $\sqrt{3}/2$, and they divide each other at a ratio of $2 : 1$, so that the center (where they all intersect) is at distance $\frac{2}{3}\sqrt{3}/2 = 1/\sqrt{3} > 1/2$ from the vertices.
- (e) Let the vertices of the hexagon be V_1, \dots, V_6 . By drawing 3 lines connecting V_i and V_{i+3} for $1 \leq i \leq 3$, we split the hexagon into 6 equilateral triangles of side length 1. So if we have at least 7 points, then by the PHP two of them are guaranteed to be within distance 1 from each other. It is not too hard to place 5 points that are of mutual distance more than 1 from each other: place one at V_3 , one each at distance ε from V_2 and V_4 , and one each at distance 2ε from V_1 and V_5 , for some small $\varepsilon > 0$.

What about 6 points? Extra credit: Even 6 points guarantee two at distance at most 1 from each other. By the above it follows that we are done unless there is exactly one point in each of the 6 triangles. Let M be the midpoint of the hexagon and let d_i be the distance of the point P_i in the triangle $V_i V_{i+1} M$ from the vertex V_i . By relabeling vertices if necessary we may assume that d_1 is the largest of the d_i , and thus $d_2 \leq d_1$. By drawing circles of radius d_1 and d_2 around V_1 and V_2 respectively it is easy to see that the farthest P_1 and P_2 can be apart is when P_2 is on $V_2 V_3$, and thus when P_1 is on $V_1 M$. By drawing a perpendicular line from P_2 to $V_1 M$ and observing that $1 \geq d_1 \geq d_2$ it follows that the distance between P_1 and P_2 is largest if $d_1 = 1$ or $d_1 = d_2$. In the first case $P_1 = M$ is in $V_2 V_3 M$ and thus at distance at most 1 from P_2 , whereas in the second case $V_1 V_2 P_2 P_1$ is a parallelogram, so that the length of $P_1 P_2$ equals that of $V_1 V_2$, which is 1.

4. (a) Creating a subset of $[n] = \{1, 2, \dots, n\}$ consists in deciding for each of the numbers in $[n]$ whether it is in the set or not. In other words a subset S corresponds to a binary n -tuple (x_1, x_2, \dots, x_n) so that $x_i = 1$ iff $i \in S$. For each x_i there are 2 choices (0 or 1) independent of the other x_j 's. Thus it follows by the generalized multiplication principle that there are $2 \cdot 2 \cdot 2 \cdots 2 = 2^n$ ways of choosing such a tuple, and thus a subset of $[n]$.
- (b) For a given $S \subseteq [n]$, let $\bar{S} = [n] - S$ denote its complement. Observe that $S \neq \bar{S}$, since only one of these sets contains the element 1. Thus we can partition the subsets of $[n]$ into sets of size 2 of the form $\{S, \bar{S}\}$. This partition has, by (a), exactly $2^n/2 = 2^{n-1}$ parts. Thus if we have a collection of $2^{n-1} + 1$ subsets of $[n]$, then by the PHP at least two of them must be from the same part of this partition $\{S, \bar{S}\}$. But since $S \cap \bar{S} = \emptyset$, these sets are disjoint as desired.
- (c) Consider the collection C of all subsets of $[n]$ which contain the element n . Since for all $S, T \in C$ we have $n \in S \cap T$, it follows that no two of these subsets are disjoint. Furthermore every $S \in C$ corresponds to a unique subset $S' = S - \{n\}$ of $[n - 1]$, so that $|C| = \#$ subsets of $[n - 1]$, which is 2^{n-1} by (a).
5. The number of four digit numbers $abcd$ of a certain type can be determined by the generalized multiplication principle, by determining the number of choices for a, b, c, d .
- (a) Use only digits 1,2,3,4,5: 5 choices for a, b, c and d each for a total of $5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$ possibilities.
- (b) (a) without repetition: 5 choices for a , 4 for b (anything but a), 3 for c and 2 for d for a total of $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possibilities.
- (c) (a) with last digit even: only 2 and 4 are even, so there are 2 choices for d , but then 5 choices each for a, b, c , for a total of $2 \cdot 5 \cdot 5 \cdot 5 = 250$ possibilities.
- (d) (b) with last digit even: 2 choices for d , 4 for a , 3 for b and 2 for c , for a total of $2 \cdot 4 \cdot 3 \cdot 2 = 48$ possibilities.
- (e) Since 2, 7 may not appear we only have 8 digits available. There are two ways of achieving such a number: either $a = 5$ and $b \geq 4$ or $a > 5$. For the first case there is 1 choice for a , 4 for b (4,6,8,9), $8 - 2$ for c and $8 - 3$ for d for a total of $1 \cdot 4 \cdot 6 \cdot 5 = 120$.
In the second case there are 3 choices for a (6,8,9), $8 - 1$ for b , $8 - 2$ for c and $8 - 3$ for d , for a total of $3 \cdot 7 \cdot 6 \cdot 5 = 630$. Altogether there are $120 + 630 = 750$ such numbers.
6. (a) The cards in the deck be ordered in $52 \cdot 51 \cdot 50 \cdots 1 = 52!$ ways, since we can choose the cards one after the other from the remaining cards.
- (b) There are 13 cards in each of the 4 suits. We first decide in which order the suits appear. There are 4 suits, so there are $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ possibilities. Next we decide for each suit in which order the cards in the suit appear. This can be done in $13!$ ways for each of the suits. Altogether we obtain $4!(13!)^4$ possible ways.
- (c) A poker hand can be dealt in $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200$ ways.
- (d) Each possible poker hand could have been dealt in one of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways (which card comes first, second, third, ...). Thus there are $311875200/120 = 2598960$ different poker hands.
7. Recall that if the prime factorization of n is $p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}$, then n has exactly $(k_1 + 1)(k_2 + 1)(k_3 + 1) \cdots (k_r + 1)$ positive divisors, since every such divisor is of the form $p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_r^{m_r}$, where $0 \leq m_i \leq k_i$.
- (a) $n = 3^4 \cdot 5^2 \cdot 7^6 \cdot 11^1$, so there are $5 \cdot 3 \cdot 7 \cdot 2 = 210$ such divisors.
- (b) $n = 620 = 2^2 \cdot 5^1 \cdot 31^1$, so there are $3 \cdot 2 \cdot 2 = 12$ such divisors.
- (c) $n = 10^{10} = 2^{10} 5^{10}$, so there are $11 \cdot 11 = 121$ such divisors.