Math 474, Spring 2004
Assignment 4

due: Thursday, February 26, 2004

1. Determine the number of *integer* solutions to the equation $x_1+x_2+x_3+x_4 = 20$ in the following situations:
   (a) There are no further restrictions on x_1, x_2, x_3, x_4.
   (b) We must have $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.
   (c) We must have $x_1 \geq 1, x_2 \geq 3, x_3 \geq -3, x_4 \geq 0$.
   (d) We must have $x_1 \leq 8, x_2 \leq 10, x_3 \leq 12, x_4 \leq 5$.
   (e) We must have $3 \leq x_1 \leq 8, 0 \leq x_2 \leq 10, -4 \leq x_3 \leq 12, -20 \leq x_4 \leq 30$.

2. We want to place (some of) 5 black rooks and 3 white rooks such that no two can attack each other.
   (a) In how many ways can the 5 black rooks be placed on an 8x8 chessboard?
   (b) (3.6.18a) In how many ways can all 8 rooks be placed on an 8x8 chessboard?
   (c) (3.6.18b) In how many ways can all 8 rooks be placed on a 12x12 chessboard?
   (d) In how many ways can some 3 of the rooks be placed on an 8x8 chessboard?
   (e) In how many ways can some 5 of the rooks be placed on an 8x8 chessboard?

3. (a) (3.6.34) In how many ways can 12 apple drinks and 1 orange drink be distributed among 3 thirsty children so that each child receives at least one drink?
   (b) (3.6.35) In how many ways can 10 orange drinks, 1 lemon drink and 1 lime drink be distributed among 4 thirsty children so that each child receives at least one drink, and the lemon and lime drinks go to different children?

4. New proofs for old formulas:
   (a) (5.8.1) Prove Pascal's formula by using the fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   (b) Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$ by using the fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   (c) Use the lattice-walk idea from lecture 8 to prove that $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$.
   (d) Use the lattice-walk idea from lecture 8 to prove that $\binom{n}{k} = \binom{n}{n-k}$.

5. Applications of the binomial theorem.

   (a) (5.8.6) Find the coefficients of x^5y^{13} and x^8y^9 in the expansion of $(3x-2y)^{18}$.

   (b) (5.8.7a) Prove that $\sum_{k=0}^{n}\binom{n}{k}2^k = 3^n$.

   (c) (5.8.7b) Generalize (b) to determine $\sum_{k=0}^{n}\binom{n}{k}r^k$ for any real number r.

   (d) (5.8.8) Evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}$.

   (e) (5.8.9) Evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}10^{-k}$ in two different ways using the binomial theorem.

6. (a) (5.8.16) Prove by integrating the binomial expansion, that for every positive integer n

$$1\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\ldots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}.$$

   (b) (5.8.17) Prove (a) by using 4.b) and 4.c).

   (c) (5.8.18) Evaluate the sum

$$1\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots+(-1)^n\frac{1}{n+1}\binom{n}{n}.$$

7. (5.8.25/26) Summations of products of binomial coefficients:

   (a) Use a combinatorial argument to prove the *Vandermonde convolution* for the binomial coefficients: for all nonnegative integers n, m_1, m_2:

$$\sum_{k=0}^{n}\binom{m_1}{k}\binom{m_2}{n-k}=\binom{m_1+m_2}{n}.$$

   (b) Derive the following identity as a special case of (a): $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$.

   (c) Let n, m_1, m_2, m_3 be fixed nonnegative integers. Find and prove a formula for $\sum\binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$, where the summation extends over all nonnegative integers r, s, t for which $r+s+t=n$.