Math 474, Spring 2004
Assignment 10

due: Thursday, April 22, 2004

1. Let l, m, n, q be natural numbers with $q \le m$.
   (a) Prove that $\displaystyle\sum_{n=0}^{\infty} \binom{n+q}{m} x^n = \frac{x^{m-q}}{(1-x)^{m+1}}$.
   (b) Use generating functions to determine $\displaystyle\sum_{k=0}^{n} \binom{q+k}{m}\binom{n-k}{l}$.
   (c) Use generating functions to determine $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{s}{k-m}\binom{n-k}{l}$ where s is any real number.

2. Suppose c, d, a_0, a_1 are constants and consider the sequence a_n determined by $a_n = ca_{n-1} + da_{n-2}$ for $n \ge 2$.
   (a) Express $A(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ as a quotient of two polynomials.
   (b) Suppose $1 - cx - dx^2 = (1-\alpha x)(1-\beta x)$ with $\alpha \ne \beta$.
   Determine constants A, B so that $A(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$.
   (c) Give a closed form for a_n using A, B, α, β.
   (d) Use the previous parts to find the generating function and closed form for a_n when $a_0 = 1$, $a_1 = 4$ and $a_n = 5a_{n-1} - 6a_{n-2}$.

3. Let a_n be given by $a_0 = 0$, $a_1 = 3/2$ and $a_n = -\frac{1}{2} + \frac{1}{2}\displaystyle\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \ge 2$.
   (a) Show that $A(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ satisfies $2A(x) = 1 + 4x - \frac{1}{1-x} + A(x)^2$.
   (b) Show that $(A(x) - 1)^2 = \frac{(1-2x)^2}{1-x}$.
   (c) Show that $A(x) = 1 - (1-2x)(1-x)^{-1/2}$.
   (d) Give a closed form for a_n.
   (e) How would a_n change for $n \ge 1$ if $a_0 = -2$?

4. (7.8.17) Let $2n$ points be given on a circle. A *segment* is any straight line segment that connects two of these points. A *matching* is a collection of n segments such that no two segments intersect or have a common endpoint. Let h_n be the number of matchings of $2n$ points.
   (a) Determine h_1, h_2, and h_3 by exhibiting all possible matchings on 2, 4 and 6 points.
   (b) Establish a recurrence relation for h_n for $n \ge 2$.
   (c) Determine h_n for all $n \ge 1$.

5. (7.8.30) You have a gigantic crate which contains infinitely many 2-packs of apples and 3-packs of bananas, as well as two oranges and one pear. Two bags of fruits are indistinguishable if they have the same number of fruits of each type, and you don't ever break open the multipacks.

   (a) Let p_n be the number of distinguishable bags you can make that contain n pears, but no other fruits. Determine $P(x) = \sum_{n=0}^{\infty} p_n x^n$. Do the same thing for oranges.

   (b) Let a_n be the number of distinguishable bags you can make that contain n apples, but no other fruits. Determine $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

   (c) Let b_n be the number of distinguishable bags you can make that have n bananas, but no other fruits. Determine $B(x) = \sum_{n=0}^{\infty} b_n x^n$.

   (d) Let h_n be the number of distinguishable bags with n fruits you can make, if you can use any of the fruits from your crate. Determine $H(x) = \sum_{n=0}^{\infty} h_n x^n$.

   (e) Find a closed formula for h_n.

6. A Fibonacci convolution. Recall that $F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\Phi x} - \frac{1}{1-\overline{\Phi}x}\right)$.

   (a) Prove that $\sum_{n=0}^{\infty} F_{n+1}x^n = \frac{1}{1-x-x^2}$.

   (b) Prove that $\sum_{n=0}^{\infty} (2F_{n+1} - F_n)x^n = \sum_{n=0}^{\infty} (\Phi^n + \overline{\Phi}^n)x^n$.

   (c) Prove that $5F(x)^2 = \sum_{n=0}^{\infty} \binom{n+1}{1} \Phi^n x^n - 2\sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} \binom{n+1}{1} \overline{\Phi}^n x^n$.

   (d) Prove that $\sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1} - (n+1)F_n}{5}$.

7. (a) (7.8.31) Determine the generating function for the number h_n of nonnegative integer solutions of $2x_1 + 5x_2 + x_3 + 7x_4 = n$.

   (b) Prove that the number of nonegative integer solutions of $2x_1 + 2x_2 + x_3 + 2x_4 + 2x_5 = n$, where x_1 is odd, $x_2 \leq 1$, x_3 is even, and $x_4 \geq 5$ is the same as the number of nonegative integer solutions to $y_1 + y_2 + y_3 + y_4 = \frac{n}{2} - 6$.