Math 474, Spring 2004

Assignment 11

due: Tuesday, May 4, 2004

1. (a) Explain the bijection between triangulations of a convex $(n+1)$-gon and multiplication schemes (that is parenthesizations) of n symbols $a_1, a_2, \dots a_n$ in your own words. That is **explain** how to get a parenthesization from a triangulation and vice versa.
   (b) (8.5.3) Write out all the multiplication schemes of four symbols, and the corresponding triangulations of a convex 5-gon.
   (c) (8.5.4) Determine the triangulation that corresponds to the parenthesization $a_1(((a_2a_3)(a_4a_5))a_6)$.
   (d) (8.5.4) Determine the triangulation that corresponds to $((a_1a_2)(a_3(a_4a_5)))((a_6a_7)a_8)$.

2. Bijections with acceptable sequences.
   (a) (8.5.2) Let n be a positive integer. A $2 \times n$ array

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \end{bmatrix}$$

   is called *increasing* if $x_{11} < x_{12} < x_{13} < \cdots < x_{1n}$, $x_{21} < x_{22} < x_{23} < \cdots < x_{2n}$ and $x_{1k} < x_{2k}$ for all $1 \le k \le n$. Show that there are C_n increasing $2 \times n$ arrays of $1, 2, 3, \dots, 2n$.
   (b) Give a bijection (that is a direct correspondence) that shows that the number of parenthesizations of $a_1a_2 \dots a_{n+1}$ is the same as the number of acceptable sequences of n $+1$'s and n -1's.

3. (8.5.5) Let m, n be integers with $n \ge m \ge 0$. Some $m+n$ people want to buy a \$5 movie ticket each. n of these people have correct change (that is \$5 bills), whereas the remaining m only have \$10 bills. Prove that these $m+n$ people can line up at the box office window in exactly $\frac{n-m+1}{n+1}\binom{m+n}{m}$ ways, so that the cashier can always give the correct change, even if she started out with an empty cash register.

4. (a) (8.5.6) Let $h_n = 2n^2 - n + 3$ for all $n \ge 0$. Determine the difference table of this sequence and find a formula for $\sum_{k=0}^{n} h_k$.
   (b) (8.5.7) Suppose h_n is a polynomial in n of degree 3 with $h_0 = 1, h_1 = -1, h_2 = 3$ and $h_3 = 10$. Give formulas for h_n and for $\sum_{k=0}^{n} h_k$.
   (c) (8.5.8) Find a closed expression for $\sum_{k=0}^{n} k^5$.

5. (a) Show that $\Delta\binom{n}{k} = \binom{n}{k-1}$ for all integers k.
   (b) (8.5.10) Suppose that p and m are integers with $p \geq m \geq 0$ and $d_0, d_1, \ldots, d_p, c_0, c_1, \ldots, c_m$ are constants such that for all $n \geq 0$

$$d_0\binom{n}{0} + d_1\binom{n}{1} + \ldots + d_p\binom{n}{p} = c_0\binom{n}{0} + c_1\binom{n}{1} + \ldots + c_m\binom{n}{m}.$$

   Show that $c_k = d_k$ for all $0 \leq k \leq m$ and $d_k = 0$ for all $m < k \leq p$.

   (c) Prove that the representation from Theorem 8.2.2 is unique.

6. (a) (8.5.9) Prove that for every sequence $h_0, h_1, h_2, \ldots$ we have $\Delta^p h_n = \sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}h_{n+k}$.
   (b) Prove that for every sequence $h_0, h_1, h_2, \ldots$ we have $\Delta^p h_n = \sum_{k=0}^{n}\binom{n}{k}\Delta^{p+k}h_0$.

7. Let g_n, h_n be arbitrary sequences.

   (a) Prove the following "product rule": $\Delta(g_n h_n) = (g_n \Delta h_n) + (h_{n+1}\Delta g_n)$.
   (b) State and prove a "reciprocal rule" for computing $\Delta\frac{1}{g_n}$.
   (c) State and prove a "quotient rule" for computing $\Delta\frac{h_n}{g_n}$.
   (d) State and prove a formula for $\Delta(an + b)$, where a, b are constants.
   (e) Use only the rules from the previous parts to compute $\Delta\left(\frac{n^2-1}{2n+1}\right)$ and express it as the quotient of two simplified olynomials.