Math 490, Example

E.g.

$T=2,\ S_0=2,\ u=3/2,\ d=5/4,\ r=1/3\ \ p=3/5.$

X is European call option with $K=\$3.50$.

Recall that $\phi_1^*=(5/8, -9/8)$

and $$\phi_2^* = \begin{cases} (1, -63/32), & \text{if } \xi_1=u, \\ (2/5, -45/64), & \text{if } \xi_1=d. \end{cases}$$

Let $c_0=C_0$, $c_1^u=C_1(\omega_1)=C_1(\omega_2)$ and $c_1^d=C_1(\omega_3)=C_1(\omega_4)$.

(1) Suppose that $c_0=1$.

Show that an arbitrage opportunity exists.

Let $\psi_1=(5/8, -2/8, -1)=(\phi_1^*, \beta_1^*+7/8, -1)$

and $$\psi_2=(\phi_2^*, \beta_2^*+7/8, -1)=\begin{cases} (1, -35/32, -1), & \text{if } \xi_1=u, \\ (2/5, 11/64, -1), & \text{if } \xi_1=d. \end{cases}$$

Then
$$V_1(\psi)=\begin{cases} \frac{5}{8}\cdot 3-\frac{2}{8}\frac{4}{3}-c_1^u=37/24-c_1^u, & \text{on } \xi_1=u, \\ \frac{5}{8}\frac{5}{2}-\frac{2}{8}\frac{4}{3}-c_1^d=59/48-c_1^d, & \text{on } \xi_1=d. \end{cases}$$

and

$$\alpha_2 S_1+\beta_2(1+r)+\gamma_2 C_2=\begin{cases} 3-\frac{35}{32}\cdot\frac{4}{3}-c_1^u=\frac{37}{24}-c_1^u, & \text{on } \xi_1=u, \\ \frac{2}{5}\cdot\frac{5}{2}+\frac{11}{64}\cdot\frac{4}{3}-c_1^d=\frac{59}{48}-c_1^d, & \text{on } \xi_2=d. \end{cases}$$

So ψ is self-financing.

Next we verify that ψ is an arbitrage opportunity. For this note that

(A) $V_0(\psi) = \frac{5}{8}2 - \frac{2}{8} - 1 = 0$

(B)
$$V_2(\psi) = \begin{cases} 1 \cdot \frac{9}{2} - \frac{35}{32} \cdot \frac{16}{9} - 1 = \frac{14}{9}, & \text{on } \xi_1 = \xi_2 = u, \\ 1 \frac{15}{4} - \frac{35}{32} \cdot \frac{16}{9} - \frac{1}{4} = \frac{14}{9}, & \text{on } \xi_1 = u, \xi_2 = d, \\ \frac{2}{5} \cdot \frac{15}{4} - \frac{11}{64} \cdot \frac{16}{9} - \frac{1}{4} = \frac{14}{9}, & \text{on } \xi_1 = d, \xi_2 = u, \\ \frac{2}{5} \cdot \frac{25}{8} - \frac{11}{64} \cdot \frac{16}{9} - 0 = \frac{14}{9}, & \text{on } \xi_1 = d, \xi_2 = d. \end{cases}$$

So $V_2(\psi) = 14/9 > 0$

Then (i) $V_2(\psi) \geq 0$ and (ii) $P(V_2(\psi) > 0) = 1 > 0$.

So ψ is an arbitrage opportunity.

(2) Suppose that $\alpha_1^u = 1/4$.
Show that an arbitrage opportunity exists.

Idea: Do nothing at time 0.
Do nothing at time 1 if $\xi_1 = d$.
Do something that is self-financing at time 1 if $\xi_1 = u$.

So $\psi_1 = (0,0,0)$

$$\psi_2 = \begin{cases} \text{something}, & \text{if } \xi_1 = u, \\ (0,0,0), & \text{if } \xi_1 = d. \end{cases}$$

You need to take advantage of the fact that the claim is underpriced at time 1 on $\xi_1 = u$, and since $V_1(\psi) = 0$, you need to do this in a way that costs ~~on~~ a net value of zero dollars.

$\boxed{\text{Homework 4 Question 1: Define } \psi_2^u \text{ so that } \psi \text{ is an arbitrage opportunity.}}$