Math 540, Fall 2004
Final Exam

due: Tuesday, December 14, 2004, 4PM

Do all questions !!

1. Explain the main results of the 75 minute presentation given by one of your classmates *in your own words.* You do not need to give any proofs. Your write-up should be about 2 pages, and you should emphasize what *you* found interesting/surprising. In other words, write as if you were to advertise the talk of your classmate to the unfortunate people who had to miss it.

2. Let $(P, \leq)$ be a ranked poset with rank function r and levels $S_1, S_2, \ldots, S_h$ in which every maximal chain meets every level. For each $x \in S_k$, let $N_x = |S_k|$ be the number of elements on the same level as x. Prove that the following are equivalent

   (a) P has a chain cover C with maximal chains, such that each $x \in P$ is in exactly $|C|/N_x$ of the chains.

   (b) For each antichain $\mathcal{A}$, $\sum_{x \in \mathcal{A}} 1/N_x \leq 1$.

   (c) For each $1 \leq k \leq h$ and each $A \subset S_k$ the number of elements in S_{k+1} that cover elements in A is at least $|A| \frac{|S_{k+1}|}{|S_k|}$.

   (Hint: for (c)$\Rightarrow$(a) let $M = |S_1||S_2| \ldots |S_h|$ and obtain a poset P' by replacing each $x \in P$ by M/N_x copies of x with the same comparisons as the original x.)

3. (a) Determine the maximum number of colors k such that every k-coloring of the subsets of $[n]$ must yield distinct sets A, B such that $A, B, A \cap B, A \cup B$ all have the same color.

   (b) Determine the maximum number of colors k such that every k-coloring of the subsets of $[6]$ must yield incomparable sets A, B such that $A, B, A \cap B, A \cup B$ all have the same color.

4. Let $A_n = (a_{ij})$ denote the cyclic Latin square of order n, where $a_{ij} = i + j - 1$ modulo n. Prove that there is a latin square which is orthogonal to A_n if and only if n is odd.

5. Bad resolution?

   (a) Determine all resolvable symmetric designs up to isomorphism.

   (b) Show that you can't find triples $D_1, D_2, \ldots, D_m \subseteq Z_v$ such that $\{D_i + j : 1 \leq i \leq m, 0 \leq j \leq v - 1\}$ (where addition is modulo v) forms a resolvable design with vm distinct blocks and $r + v \leq 35$.

6. Let $\mathbf{C}$ be a perfect linear (n, k)-code of minimum distance 3.

   (a) Show that $(i + 1)A_{i+1} = \binom{n}{i} - A_i - (n - i + 1)A_{i-1}$ for all $1 \leq i \leq n - 1$.

   (b) The *support* of $x \in \mathbf{C}$ is $\{i : x_i = 1\}$. Prove that the support of all codewords of weight 3 forms a Steiner triple system.

   (c) Show that $\mathbf{C}$ must contain the codeword of weight n.

   (d) Show that $A_{n-i} = A_i$ for all $0 \leq i \leq n$.