Methods of Hadamard Matrix Construction

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Math 540

<u>Definition:</u> if H is a matrix of $+1, -1$ of order $n \times n$ and $H \cdot H^T = nI$ then H is called a <u>Hadamard Matrix.</u>

- Currently it has been shown that Hadamard matrices exist for orders $1, 2, 4 \ldots, 4n$ for n up to 107.

- While it is not Known if Hadamard matricies exist for all $n \geq 1$ $4n$, we do know of several infinite families of Hadamard matricies.

- We have seen the form $H_2 = \begin{pmatrix} + & + \\ + & - \end{pmatrix}$ and

$$H_2 \otimes H_2 = H_4 = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}$$

This can be expanded to 2^α $\alpha \geq 1$

- In general $H_n \otimes H_m \Rightarrow H_{nm}$
(Theorem 18.4 - This result was shown in class)

- We've also seen Williamson's method

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & A_1 & -A_2 \\ -A_4 & -A_3 & A_2 & A_1 \end{pmatrix}$$

where $A_1^2 + A_2^2 + A_3^2 + A_4^2 = 4nI$
- Symmetric
- Commute

(this was also shown in class)

Part 1   Hadamard - Design Construction Method

- A (v, k, λ) design is called symmetric when the # of blocks (b) is equal to the # of elements (v).

- It can be shown that when $n = k - \lambda$ there are bounds for v [redacted] $4n-1 \le v \le n^2+n+1$ when symmetric
(We have seen that $\lambda = 1$ [redacted] $v = n^2+n+1$ is a finite projective plane.)
Lets examine the designs where $v = 4n-1$ (The Lower Bound)

<u>Theorem:</u> Any symmetric design with $v = 4n-1$ $n = k-\lambda$ is either a $(4n-1, 2n-1, n-1)$ design or its complement.

<u>Proof:</u> Recall from class lecture $r = \dfrac{\lambda(v-1)}{k-1}$ and $vr = kb$
since $v = b$ (in a symmetric design) this implies that $r = k$

$$\Rightarrow \quad r(k-1) = \lambda(v-1) \quad \Rightarrow \quad k(k-1) = \lambda(4n-2) = 2\lambda(2n-1)$$

$$n = (k-\lambda) \Rightarrow \lambda = (k-n) \qquad = 2(k-n)(2n-1)$$

$$= 4kn - 4n^2 + 2n - 2k$$

$$= 4kn - 2k - 4n^2 + 2n$$

$$k(k-1) = k(4n-2) - 2n(2n-1)$$

$$\Rightarrow \quad 0 = k^2 - k - k(4n-2) + 2n(2n-1)$$

$$= k^2 - k - 4kn + 2k + 4n^2 - 2n$$

$$= k^2 - 2kn - 2kn + 4n^2 + k - 2n$$

$$= k(k-2n) - 2n(k-2n) + (k-2n)$$

$$0 = (k-2n)(k-2n+1) \quad \Rightarrow \quad k = 2n \text{ or } k = 2n-1$$

• If $K = 2n$ then since $n = k - \lambda \Rightarrow \lambda = n$ and the result is a $(4n-1, 2n, n)$ design.

• If $K = 2n+1$ then since $n = k - \lambda \Rightarrow \lambda = n-1$ and the result is a $(4n-1, 2n-1, n+1)$ design.

• Verify they are complements $(4n-1) - (2n) = (2n-1)$
• for design (v, k, λ) a complement is $(v, v-k, \lambda')$ where $\lambda' = b - 2r + \lambda$. This holds in our case above.

<u>Definition</u>: A $(4n-1, 2n-1, n-1)$ Symmetric Design is called a <u>Hadamard Design</u> of order n.

<u>Lemma</u>: If H is a Hadamard matrix, then $H \cdot H^T = H^T \cdot H = n I$

<u>proof</u>: recall from linear algebra that $A \cdot A^{-1} = A^{-1} A$ where A and A^{-1} are $n \times n$ matrices. So what is H^{-1}? $H^{-1} = \frac{1}{n} H^T \Rightarrow H \cdot H^{-1} = H \cdot \frac{1}{n} H^T = I$.
but this implies $H \cdot H^{-1} = H^{-1} \cdot H = \frac{1}{n} H^T \cdot H = I$
Since $\frac{1}{n}$ is just a scalar $H \cdot H^T = n I = H^T \cdot H$. ∎
This implies that all Hadamard Matricies are <u>both</u> row orthogonal as well as column orthogonal.

• Also Recall that if the 1^{st} row of H is normalized (all +'s) a matrix of order m then each row (apart from the 1^{st}) has $\frac{1}{2}m$ +'s and $\frac{1}{2}m$ −'s. Furthermore there will be $\frac{1}{4}m$ +'s that match and $\frac{1}{4}m$ −'s that match (giving inner product zero)

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- Given a Hadamard Design of order n take the incidence Matrix A and replace every 0 by $-$, then add a further row and column of $+$'s. Call this $(+,-)$ Matrix H.

<u>Theorem 1</u>: If A is the incidence matrix of a Hadamard design of order n $(4n-1, 2n-1, n-1)$ then H is a Hadamard matrix of order $4n$.

Proof: Consider the dot product of any two columns of H. Since $\lambda = (n-1)$ each column will have $+$'s together $(n-1)+1 = n$ times (λ + row of $+$'s added). Also, since $r = K$ in a symmetric design each column contains $r+1 = (2n-1)+1 = 2n$ $+$'s. (and $2n$ $-$'s). So there are $2(2n-n) = 2n$ places they differ (product = -1) Then $-$'s will match up n times and the dot-product $2n(+) + 2n(-) = 0$ $\Rightarrow$ H is a Hadamard matrix of order $4n$. Since each column beyond the 1^{st} has $2n$ $+$'s and $2n$ $-$'s the dot-product with the 1^{st} column and any other will also be 0. ∎

<u>Theorem 2</u>: Given a Hadamard matrix H of order $4n$ A will be the incidence matrix of a $(4n-1, 2n-1, n-1)$ Hadamard Design.

<u>Proof</u>: After normalizing H in the first row and column and removing the first row and column A will result.