Math 540, Fall 2004
Assignment 2

due: Tuesday, October 5, 2004

All sets in this problem set are finite.

1. Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq [n]$ be a k-set with $a_1 < a_2 < \ldots < a_k$.

   (a) Prove that $|I_k(A)| = \binom{a_k - 1}{k} + \binom{a_{k-1} - 1}{k-1} + \cdots + \binom{a_1 - 1}{1} + 1$.

   (b) Fix an integer $k > 0$. Prove that every integer m can be uniquely written as $m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_i}{i}$ with $m_k > m_{k-1} > m_{k-2} > \ldots > m_i \geq i$.

   (c) Use the decomposition from (b) to find the $(m+1)$-st set in the colex order of $\binom{[n]}{k}$, as well as the minimum size of the shadow of a collection of $m+1$ k-sets.

2. Prove that the largest antichain in B_n which consists of pairs of complementary sets has size $2\binom{n-1}{\lceil n/2 \rceil}$. (Hint: Find a related intersecting family.)

3. Let $([n], \mathcal{A})$ be a set-system and $i, j \in [n]$ be distinct. For any set A with $i \in A, j \notin A$ we let $A_{ij} = A - \{i\} + \{j\}$. The *shift operator* τ_{ij} is defined by $\tau_{ij}(A) = A_{ij}$ provided that $i \in A, j \notin A$, and $A_{ij} \notin \mathcal{A}$; otherwise $\tau_{ij}(A) = A$. With this notation we have $\tau_{ij}(\mathcal{A}) = \{\tau_{ij}(A) : A \in \mathcal{A}\}$.

   (a) Prove that $|\tau_{ij}(\mathcal{A})| = |\mathcal{A}|$.

   (b) Prove that if $\mathcal{A}$ is intersecting, then so is $\tau_{ij}(\mathcal{A})$.

   (c) Prove that $\partial(\tau_{ij}(\mathcal{A})) \subseteq \tau_{ij}(\partial \mathcal{A})$.

4. Let t, k be different natural numbers, and $\mathcal{A}$ be a family of k-sets of $[n]$. We define the *t-shadow* of $\mathcal{A}$, denoted by $\partial_t(\mathcal{A})$ as follows: if $t < k$, then $\partial_t(\mathcal{A})$ consists of all t-element subsets of sets in $\mathcal{A}$; if $t > k$, then $\partial_t(\mathcal{A})$ consists of all t-element subsets of $[n]$ which have sets in $\mathcal{A}$ as subsets.

   (a) Suppose that $t < k$. Show that the t-shadow of a family of m elements of $\binom{[n]}{k}$ is minimized by taking the first m elements in the colex order of $\binom{[n]}{k}$.

   (b) Suppose that $t > k$. Describe a family of m elements of $\binom{[n]}{k}$ whose t-shadow is as small as possible. Prove your answer.

5. For a family $\mathcal{A}$ of subsets of $[n]$, let n_k be the number of k-sets in $\mathcal{A}$, and the *profile* of $\mathcal{A}$ be the sequence $n_0, n_1, n_2, \ldots, n_n$. Also, let $\partial_k(a)$ denote the minimum size of the shadow of a family of exactly a sets of size k, as investigated in 1(c).

   (a) Prove that the sequence $0, \ldots, n_k, n_{k+1}, \ldots, n_l, 0, \ldots, 0$ is the profile of an antichain if and only if

   $$n_k + \partial_{k+1}(n_{k+1} + \partial_{n+2}(n_{k+2} + \ldots + \partial_{l-1}(n_{l-1} + \partial_l(n_l))\ldots)) \leq \binom{n}{k}.$$

   (b) Suppose $\mathcal{A}$ is a family of k-sets of $[n]$ and $\mathcal{B}$ is a family of l-sets of $[n]$ with $k + l \leq n$. Use (a) to show that if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}, B \in \mathcal{B}$, then either $|\mathcal{A}| < \binom{n-1}{k-1}$ or $|\mathcal{B}| \leq \binom{n-1}{l-1}$.

   (c) Use (b) to give an easy proof of the Erdős-Ko-Rado theorem.

6. (a) Prove that if $\mathcal{A}$ is an intersecting family of k-sets from $[n]$, then $|\partial\mathcal{A}| \geq |\mathcal{A}|$. (Hint: for $k \leq n/2$, use induction on n and k with basis step $k = 2$. For the induction step apply $\tau_{i,n}$ described in problem 3.)

   (b) Show that if $\mathcal{A}$ is an intersecting family of k-sets from $[n]$, then the complements of sets in $\mathcal{A}$ also form an intersecting family as long as $k \leq \lfloor n/2 \rfloor$.

   (c) Let $q = \lfloor n/2 \rfloor + 1$. Prove that the maximum size of an intersecting antichain of subsets of $[n]$ is $\binom{n}{q}$. (Hint: use previous parts to eliminate sets on fewer than q elements.) This is Milner's theorem (1968).