Assignment 2

due: Thursday, February 16, 2006

1. (a) (3.1.30b) For every natural number k, inductively construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ so that the greedy coloring with respect to σ uses $k+1$ colors.

   (b) (3.1.34) Show that the graph obtained from K_3 by attaching a leaf to each one of the 3 vertices has a 3-coloring in which each color is used exactly twice (and that this coloring is unique up to symmetry). Prove that such a coloring cannot be produced by any greedy coloring.

2. (3.1.32) Suppose that G has no induced subgraph isomorphic to P_4. Prove that for every vertex ordering the greedy coloring procedure produces an optimal coloring (Hint: Suppose the algorithm uses k colors on the ordering $v_1, v_2, \ldots, v_n$, and let i be the smallest integer such that G has a clique with colors $i, i+1, \ldots, k$. Prove that $i = 1$.)

3. (3.1.19) Place n points on a circle where $n \geq k(k+1)$.Let $G_{n,k}$ be the $2k$-regular graph obtained by joining each point to the k nearest points in each direction on the circle. For example $G_{n,1} = C_n$. Prove that $\chi(G_{n,k}) = k+1$ if $k+1$ divides n and $\chi(G_{n,k}) = k+2$ otherwise. Prove that the lower bound on n cannot be weakened by proving that $\chi(G_{k(k+1)-1,k} > k+2$ if $k \geq 2$.

4. (3.1.45) Prove that the two 4-regular graphs displayed on page 237 are 4-chromatic. (Hint: Consider the maximum stable sets.)

5. (3.1.24) Given finite sets $S_1, S_2, \ldots, S_m$ define a graph G with $V(G) = S_1 \times S_2 \times \cdots \times S_m$ in which two vertices are adjacent if and only if they differ in every coordinate. Determine $\chi(G)$ and $\chi(\overline{G})$.

6. (3.1.21) Let G be the *unit-distance graph* in the plane: $V(G) = \mathbf{R}^2$ (so this is an infinite graph) and two vertices are adjacent precisely when the (Euclidean) distance between them is 1. Prove that $4 \leq \chi(G) \leq 7$. (Hint: for the upper bound, present an explicit coloring by regions, with attention to their boundaries.)